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THE TOREADOR PROBLEM†

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A PROBLEM of approach under noisy conditions is considered. The system consists of three objects in a plane. A stationary object creates noise, producing a biased estimate of its true coordinates. This biased estimate plays the role of the second object (the evader). The third object—the pursuer—produces a pursuit strategy based on a given rule. The first object's task is to use its ability to produce a biased estimate of its true coordinates (of the second object) so as to keep the third object beyond a given distance from its true position.

The system is considered over a fixed time interval. It is assumed that the biased estimate of the coordinates tends in time to the first object's true position; this may be interpreted, for example, as the third object's ability to gauge more accurately the first object's true position by accumulating information from its observations. It is assumed that the second object is controllable, but there are given geometric restrictions on the intensity of the control (i.e. on the controller's resources). The third object implements a direct pursuit method with given intensity.

The problem is equivalent to the situation of an immobile toreador (the first object) with a cloak (the second object), being attacked by a bull (the third object).

The problem is solved by methods of the theory of differential games [1, 2]. The basis of these methods is a positional extremal construction, which consists essentially in constructing what is known as a positional absorption set (a maximum stable bridge). This set has a stability property which means that the movements of the control system are kept within a stable bridge. Certain algorithms [3–10], similar to dynamic programming procedures [3–8], are used for the approximate construction of a stable bridge. Two constructions are used to obtain resolving positional strategies. The first (control with a guide) uses information about only one stable set, constructing a control that steers the motion over the bridge by what is known as "extremal aiming" at the motion of a certain auxiliary system (guide) moving on the bridge. The second construction (the "maximum diversion" procedure) uses information about a system of stable sets (the Lebesgue sets of the optimal guaranteed result function). The construction of a resolving control is based on extremal aiming from the current position to the nearest point of a system of stable bridges not containing the current position.

1. STATEMENT OF THE PROBLEM

Three objects are considered in a plane. The first, stationary, object is at the origin. The second object is interpreted as a virtual image of the first, created by (information-theoretic) noise. Its position is defined by a vector $x(t)=(x_1(t),\ x_2(t))$. The third object is capable of receiving information about the position of the second object only, pursuing it according to a simple pursuit rule. Its motion is described by a vector $y(t)=(y_1(t),\ y_2(t))$. The system of three

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objects is considered over a time interval $[0, T]$, $T > 0$. The behaviour of the second object, the evader, is heuristically modelled by a second-order differential equation, which qualitatively describes the information-dynamical processes in the control system

$$\dot{x}(t) = -(A(t) + D(t))x(t), \quad x(0) = x_0 \tag{1.1}$$

where $A(t)$ and $D(t)$ are 2×2 square diagonal matrices with diagonal elements $t^2 + t + 1$ and $(t^2 + t + 1)^{-1}$, respectively. The matrix $A(t)$ describes a progressively more accurate estimation of the first object's true position, accomplished by accumulating information from data observed as time passes. Its diagonal elements represent the non-linearity of the dynamical characteristics of the information component in the control system. This improved estimation process is hindered by the effect of noise on the information component, represented in (1.1) by the matrix $D(t)$, whose diagonal elements simulate the non-linearity of the dynamical characteristics of the effect of noise on the information component.

Thus, the dynamics $x(t)$ of the first object, as described by Eq. (1.1), is essentially a biased estimate of the first object's coordinates, which tends in time to the origin—the first object's true position.

The behaviour of the third object (the pursuer) is described by a vector differential equation

$$\dot{y}(t) = -\mu(T - t)(y(t) - x(t))/\| y(t) - x(t) \| + u(t) \tag{1.2}$$

$$y(0) = y_0$$

The first time on the right means that the pursuer $y(t)$ is guided toward the object $x(t)$ by a rule of simple pursuit with intensity $\mu(T - t)$. The second term characterizes the first object's ability to control the position of its virtual image (the second object). It is assumed that the control $u(t)$ obeys certain geometric constraints $\| u(t) \| < \nu$.

The problem may now be stated as follows. The first object is stationary and located at the origin. The second object (the evader) $x(t)$ moves from its initial position x_0 to the origin, its motion governed by Eq. (1.1). The third object (the pursuer) $y(t)$ moves toward the evader $x(t)$ according to a rule of simple pursuit. It is required to find the set of all initial positions y_0 of the pursuer for which a positional control $u = u(t, x, y)$ exists that will divert the pursuer $y(t)$ from the first object over a time interval $[0, T]$; the first object is represented by a circle of radius R about the origin. In other words, we have to solve a synthesis problem—to determine the initial positions y_0 and a control $u = u(t, x, y)$ that will keep the motion $y(t)$ of system (1.2) with initial condition $y(t_0) = y_0$ in the complement to the above circle. This may be classified as a positional control problem for a non-linear system with non-convex phase constraints.

2. CONSTRUCTION OF STABLE BRIDGES

The basis of our method for solving this problem is certain algorithms for the numerical solution of non-linear control problems and differential games [9, 10], which belong to the theory of positional differential games [1, 2]. The idea of the algorithms is a positional extremal construction based on the notion of a stable bridge. The numerical construction of a stable bridge uses a certain retrograde procedure [2, 9, 10].

At each stage of the retrograde procedure one has to compute set unions and intersections, which are quite complicated operations from the computational standpoint. Programs, recently devised at the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, are available to perform these operations in a plane. They have been used to solving various non-linear problems of game theory and control theory. In particular, they have made it possible to solve the problem considered in this paper.

The rigorous definition of stability for sets is as follows.

Consider a control system in an interval $[0, T]$

$$\begin{aligned} &\dot{z} = f(t,z,u,\upsilon), \quad z(0)=z_0 \\ &f(t, z, u, \upsilon) = \varphi(t, z) + B(t, z)u + C(t, z)\upsilon \\ &z \in R^n, \quad u \in P \subset R^p, \quad \upsilon \in Q \subset R^q. \end{aligned} \tag{2.1}$$

Here $z(t)$ is the phase vector of the system, u is the vector of controls, υ is the noise vector, and P and Q are convex compact sets.

It is assumed that f: $(0, T] \times R^n \times P \times Q \to R^n$ is continuous, satisfies a Lipschitz condition with respect to the phase vector and is such that the solutions of system (2.1) can be extended to the whole of $[0, T]$.

Let M be a given closed set in the phase space R^n and N a given closed set in the position space $[0, T] \times R^n$.

We wish to construct a positional control procedure $U = U(t, z)$ guaranteeing that the trajectories of system (2.1) will end up in the target set M without violating the phase constraints N: $x(t) \in M$ for some $t \in [0, T]$, $x(\tau) \in N$ for all $\tau \in [0, T]$. In this context the noise $\upsilon \in Q$ may have a negative effect.

Definition. A closed set $W \subset [0, T] \times R^n$ is called a u-stable bridge if the following conditions hold: (1) $W \subseteq N$, (2) $W(T) \subseteq M$, (3) for any positions $(t_*, x_*) \in W$ times $t^* \in (t_*, T]$ and vectors $\upsilon \in Q$, at least one of the following two relations holds:

$$X_\upsilon(\tau, t_*, x_*) \cap M \neq \varnothing \text{ for some } \tau \in [t_*, t^*]$$

$$X_\upsilon(t^*, t_*, x_*) \cap W(t^*) \neq \varnothing$$

where $W(t)$ is the set

$$W(t) = \{w \in R^n: (t, w) \in W\}$$

and $X_\upsilon(s, t_*, x_*)$ is the reachable domain at the time s of the differential inclusion

$$\begin{aligned} &\dot{x} \in F_\upsilon(t, x), \quad x(t_*) = x_* \\ &F_\upsilon(t, x) = co\{f(t, x, u, \upsilon): u \in P\} \end{aligned}$$

It is known that if the initial position is $(t_*, x_*) \in W$, then there is a control procedure with guide U that solves the approach problem with target set M, while keeping the trajectories in the set N. Intuitively, the procedure implies that, together with the motion defined by (2.1), one considers the auxiliary motion of a "guide" confined within the bridge W. In parallel one implements a control $u \in P$ that "aims" the motion of the system at the guide. Properties 1–3 of the bridge W guarantee that the motion of system (2.1) will be steered with its guide to the target M, simultaneously obeying the phase constraints N. The main difficulty in implementing the procedure is to construct a stable bridge—generally speaking, only an approximation is possible. An algorithm recently developed to compute an approximate bridge is essentially a retrograde discrete scheme. The construction of a control with a guide, however, evolves in real time, progressively using information about the constructed bridge W.

A brief description of the algorithms now follows.

Let $\Gamma = \{t_0 = 0, t_1, \ldots, t_N = T\}$ be a partition of the interval $[0, T]$, M^α a polyhedron approximating the target set M and $N^\alpha(t_i)$ a polyhedron approximating the phase constraints $N(t_i)$, $t_i \in \Gamma$.

The retrograde procedure to construct a system of sets $\{W^\alpha(t_i): t_i \in \Gamma\}$ approximating a bridge W is defined as follows:

$$W^\alpha(t_N) = M^\alpha \cap N^\alpha(t_N) \tag{2.2}$$

$$W^{\alpha}(t_i)=\pi(t_i,t_{i+1},W^{\alpha}(t_{i+1}))\cup M^{\alpha}\cap N^{\alpha}(t_i)$$
$$\pi(t_i,t_{i+1},W^{\alpha}(t_{i+1}))=\bigcap_j\bigcap_k Y^{\alpha}_{j,k}(t_i,t_{i+1},W^{\alpha}(t_{i+1}))$$

where $Y^{\alpha}_{j,k}(t_i, t_{i+1}, W^{\alpha}(t_{i+1}))$ is a polyhedral approximation to the set

$$X^{\alpha}_{j,k}(t_i,t_{i+1},W^{\alpha}(t_{i+1}))=\{w\in R^n: w+(t_{i+1}-t_i)f^{j,k}\in W^{\alpha}(t_{i+1})\} \tag{2.3}$$
$$f^{j,k}=\varphi(t_i,w)+B(t_i,w)u^k+C(t_i,w)\upsilon^j$$

u^k and υ^j are the vertices of polyhedra P^{α} and Q^{α}, respectively, that approximate the compact sets P and Q.

The operator $\pi(\cdot)$ defined by the last formula of (2.2) approximately expresses the last property of stability; we therefore call it the stable absorption operator [10].

The bulk of the work involved in implementing (2.2) consists in computing unions and intersections of polyhedra of complicated structure in R^n. Algorithms at present available can deal with unions and intersections in R^2. The retrograde procedure (2.2) for the approximate computation of a stable bridge is therefore applicable to a two-dimensional control system (1.2), whose trajectories must remain in the complement to a circle of radius R about the origin.

The retrograde procedure (2.2) for the toreador problem is as follows. The role of the non-linear function φ of (2.1) in the control system (1.2) is played by the first term on the right, where $x(t)$ is a solution of (1.1). The matrix B is in this case the 2×2 identity matrix E. The set P is the disk $B(0, \nu)$ of radius ν about the origin. There is no noise vector υ in system (1.1). To fix our ideas, we will assume that C is the identity matrix and let Q be the set consisting of one vector—the zero vector, $Q=0$. There is no target set M in this problem; to fix our ideas, we shall assume that M is the whole plane R^2. The set N of phase constraints is the Cartesian product of the interval $[0, T]$ and the closed complement $\overline{B}(0, R)$ to the circle $B(0, R)$ of radius R about the origin.

With this notation, the formulae for the retrograde procedure to construct a system of sets $\{W^{\alpha}(t_i)$: $t_i\in\Gamma\}$ approximating a stable bridge W in the toreador problem may be written as follows:

$$W^{\alpha}(t_N)=\overline{B}(0,R) \tag{2.4}$$
$$W^{\alpha}(t_i)=\pi(t_i,t_{i+1},W^{\alpha}(t_{i+1}))\cup\overline{B}(0,R)$$
$$\pi(t_i,t_{i+1},W^{\alpha}(t_{i+1}))=\bigcap_k Y^{\alpha}_k(t_i,t_{i+1},W^{\alpha}(t_{i+1}))$$

where $Y^{\alpha}_k(t_i, t_{i+1}, W^{\alpha}(t_{i+1}))$ is a polyhedral approximation to the set $X^{\alpha}_k(t_i, t_{i+1}, W^{\alpha}(t_{i+1}))$, which is analogous to the set (2.3) with the index j, k replaced by K, and

$$f^k=-\mu(T-t_i)(w-x(t_i))/\|w-x(t_i)\|+u^k,$$

$x(t_i)$ is the position at time t_i of the trajectory $x(t)$ of Eq. (1.1), u^k are the vertices of a polyhedron P^{α} approximating the geometric resource P of the control—the circle $B(0, \nu)$ of radius ν about the origin.

3. RESOLVING PROCEDURE OF CONTROL WITH A GUIDE

We shall now describe a procedure for controlling the motion $y(t)$ of system (1.2) with a guide $y^*(t)$, using recurrence relations. This control procedure is based on the stability property of a system of sets $\{W^{\alpha}(t_i): t_i\in\Gamma\}$, approximating a stable bridge W.

Define

$$y^*(t_0)=y(t_0)\in W^{\alpha}(t_0) \tag{3.1}$$

Over the interval $[t_i, t_{i+1}]$, the guide $y^*(t)$ and trajectory $y(t)$ of system (1.2) are constructed by the following rule. Compute the vectors

$$
\begin{aligned}
&s(t_i) = y^*(t_i) - y(t_i) \\
&u^*(t_i) = argmax_{u^k \in P^\alpha}\langle s(t_i), u^k \rangle
\end{aligned}
\tag{3.2}
$$

The position of the guide $y^*(t_{i+1})$ is determined from the condition

$$
\begin{aligned}
&y^*(t_{i+1}) \in \{y^*(t_i) + (t_{i+1} - t_i)F(t_i, y^*(t_i))\} \cap W^\alpha(t_{i+1}) \neq \emptyset \\
&F(t_i, y^*(t_i)) = \{f \in R^2 : f = \mu(T - t_i)(y^*(t_i) - x(t_i))/\|y^*(t_i) - x(t_i)\| + u, \quad u \in P^\alpha\}
\end{aligned}
\tag{3.3}
$$

That the intersection in (3.3) is not empty is guaranteed by the last relation in (2.4), which expresses the stability property of the system of sets $\{W^\alpha(t_i):\ t_i \in \Gamma\}$.

The trajectory $y(t)$ of system (1.2) over the interval $[t_i,\ t_{i+1}]$ is computed using the formula

$$
\begin{aligned}
&y(t) = y(t_i) + (t - t_i)(-\mu(T - t_i)(y(t_i) - x(t_i))/\|y(t_i) - x(t_i)\| + u^*(t_i)), \\
&t \in [t_i, t_{i+1}]
\end{aligned}
\tag{3.4}
$$

By the last relation of (2.4), the procedure (3.1)–(3.4) of control with a guide keeps the trajectory $y(t)$ of system (1.2) in an ε-neighbourhood of the approximating system of sets $\{W^\alpha(t_i):\ t_i \in \Gamma\}$; consequently, by the first two relations of (2.4) the trajectory remains in the ε-neighbourhood of the phase constraints $N = [0,\ T] \times \overline{B}(0,\ R)$, where ε is a number that tends to zero together with the mesh $\Delta = \max_i(t_{i+1} - t_i)$ of the partition. The procedure solves the problem of subordinating the motion $y(t)$ of the non-linear system (1.2) to the phase constraints N.

4. EXTREMAL AIMING AT A SYSTEM OF STABLE BRIDGES

We will now consider the construction of a stable bridge for system (1.2) with phase constraints $N = [0,\ T] \times \overline{B}(0,\ R)$ but now we shall vary the radius R of N.

Consider a given interval $[R_0,\ R_L]$, $R_L > R_0$, and a partition $\Gamma_R = \{R_0,\ R_1, \ldots,\ R_L\}$ of $[R_0,\ R_L]$. For each radius R_l in Γ_R, construct a system of sets $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma\}$ approximating a stable bridge W. The result will be a system of approximate stable bridges $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma,\ R_l \in \Gamma_R\}$.

The control procedure may now be defined as follows. Let us assume that the trajectory $y(t)$ of system (1.2) at time t_i satisfies the relations

$$
\begin{aligned}
&y(t_i) \in W^\alpha(t_i, R_l), \quad y(t_i) \notin W^\alpha(t_i, R_{l+1}) \\
&R_{l+1} > R_l
\end{aligned}
\tag{4.1}
$$

Find the point $w^*(t_i) \in W^\alpha(t_i,\ R_{l+1})$ closest to the point $y(t_i)$

$$
w^*(t_i) = argmin_{w \in W^\alpha(t_i, R_{l+1})}\|w - y(t_i)\|
\tag{4.2}
$$

Define the "aiming vector"

$$
s(t_i) = w^*(t_i) - y(t_i)
\tag{4.3}
$$

The resolving control $u^*(t_i)$ at time t_i is now defined by

$$
u^*(t_i) = argmax_{u^k \in P^\alpha}\langle s(t_i), u^k \rangle
\tag{4.4}
$$

The trajectory $y(t)$ of system (1.2) over the interval $[t_i,\ t_{i+1}]$ is now computed by the formula

$$y(t) = y(t_i) + (t - t_i)(-\mu(T - t_i)(y(t_i) - x(t_i))/\|y(t_i) - x(t_i)\| + u^*(t_i), \qquad (4.5)$$
$$t \in [t_i, t_{i+1}]$$

This procedure constructs a resolving control $u^*(t_i)$ on the basis of information about the entire system of nested stable sets $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma,\ R_l \in \Gamma_R\}$. At each time t_i one searches for the point $w^*(t_i) \in W^\alpha(t_i,\ R_{l+1})$ nearest to the current position $y(t_i)$ in the system of those stable sets that do not contain $y(t_i)$ (see (4.1)). Since we have assumed that $R_{l+1} > R_l$, $l = 0, \ldots, L-1$, the control procedure (4.1)–(4.5) may be called the "maximum diversion" of the third object's motion $y(t)$ from the position of the first—the origin.

If the initial position $y(t_0)$ of $y(t)$ is such that $y(t_0) \in W^\alpha(t_i,\ R_*)$ for some $R_* \in \Gamma_R$, the procedure will keep the trajectory $y(t)$ of system (1.2) in an ε-neighbourhood of the approximating system of sets $\{W^\alpha(t_i,\ R_*):\ t_i \in \Gamma\}$; consequently, by (2.4), it will also remain in an ε-neighbourhood of the phase constraints $N = [0,\ T] \times \overline{B}(0,\ R_*)$. Moreover, ε will tend to zero together with the mesh Δ of Γ. This control procedure solves the problem of subordinating the motion $y(t)$ of system (1.2) to the phase constraints $N = [0,\ T] \times \overline{B}(0,\ R_*)$.

5. RESULTS OF A COMPUTER SIMULATION

A system of sets $\{W^\alpha(t_i):\ t_i \in \Gamma\}$ approximating a stable bridge W, aimed at subordinating the trajectories of system (1.2) to phase constraints $N = [0,\ T] \times \overline{B}(0,\ R_*)$, was computed for the following special case

$$T = 1,\ \Delta = 0.01,\ \mu = 100,\ \nu = 10,\ R = 1,\ x(0) = (10;\ 0)$$
$$y(0) = y(t_0) \in W^\alpha(t_0),\quad y(0) = (-12.5;0) \qquad (5.1)$$

P^α was a regular octahedron inscribed in a circle of radius $\nu = 10$ about the origin.

Figure 1 illustrates sections at times $t_{76} = 0.75$, $t_{51} = 0.5$, $t_0 = 0$ of a system of sets $\{W^\alpha(t_i):\ t_i \in \Gamma\}$ approximating a stable bridge for the above parameter values. Each of the three sets is the complement of the compact set bounded by the curves numbered $N = 1,\ 2,\ 3$. The figure shows the trajectory $x(t)$ of system (1.1), as well as a trajectory $y(t)$ of system (1.2) implementing the extremal aiming rule. The trajectory $x(t)$ begins at the point $x(0) = (10,\ 0)$ and ends at $x(T) = (2.9,\ 0)$; it is indicated in Fig. 1 by the number 4, and its terminal point $x(T)$ is marked by an arrow. The trajectory $y(t)$ (indicated by the number 5) begins at $y(0) = (-12.5,\ 0)$ and ends at $y(t) = (3.7,\ 1.1)$. The trajectory $y(t)$ was kept by a control procedure with a guide within the phase constraints $N = [0,\ 1] \times \overline{B}(0,\ 1)$.

We also constructed a system of approximate stable bridges $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma,\ R_l \in \Gamma_R\}$ for the parameter values (5.1), in which the parameter R was varied over the interval $[R_0,\ R_L]$, $R_0 = 1$, $R_L = 4$, $L = 4$, with mesh $\Delta = 1$. Figure 2 illustrates sections at times $t_{26} = 0.25$, $t_0 = 0$ of the system $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma,\ R_l \in \Gamma_R\}$. Each of the sets $\{W^\alpha(t_i,\ R_l):\ t_i \in \Gamma,\ R_l \in \Gamma_R\}$, $t_{26} = 0.25$, $t_0 = 0$ $(l = 1,\ 2,\ 3,\ 4)$ is the

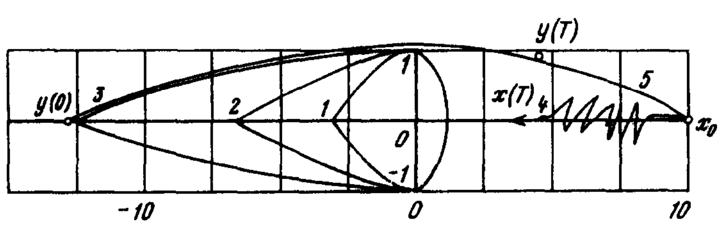

FIG. 1.

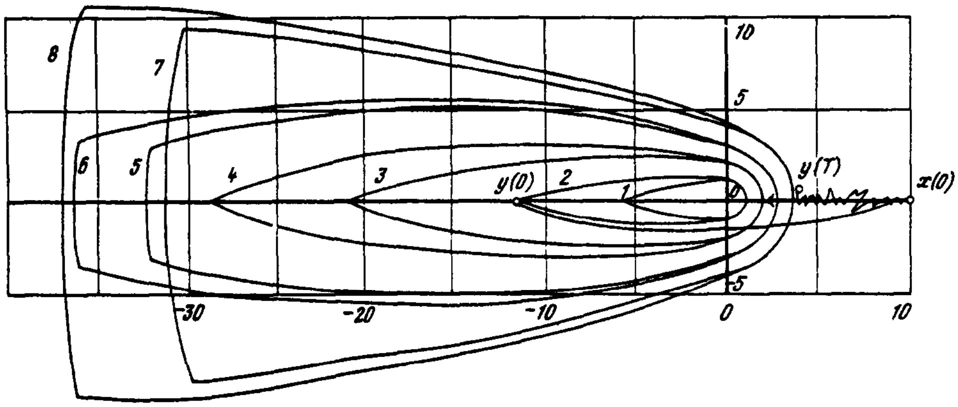

FIG. 2.

complement of the compact set bounded by the curve with the appropriate number $N=2(l-1)+j$ ($j=1$, 2, $l=1, 2, 3, 4$). We also show a trajectory $x(t)$ of system (1.1) and a trajectory $y(t)$ of system (1.2) constructed according to the "maximum diversion" procedure (4.1)–(4.5). The trajectory beings at point $x(0)=(10, 0)$ and ends at $x(T)=(2.9, 0)$; it is indicated in Fig. 2 by the number 9, and its terminal point $x(T)$ is indicated be an arrow. The trajectory $y(t)$ (indicated by the number 10) begins at $y(0)=(-12.5, 0)$ and ends at $y(t)=(3.9, 0.03)$. The trajectory $y(t)$ is kept by the "maximum diversion" procedure within the phase constraints $N=[0, 1]\times\overline{B}(0, 1)$.

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